

# On Connections between the Quantum and Hydrodynamical Pictures of Matter

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## Abstract

We present a general, model-independent, quantum statistical treatment of the connection between the quantum and hydrodynamic pictures of reservoir driven macroscopic systems. This treatment is centred on the large scale properties of locally conserved hydrodynamical observables and is designed to provide a bridge between quantum microdynamics and classical macroscopic continuum mechanics, rather than a derivation of the latter from the former. The key assumptions on which the treatment is based are hypotheses of chaoticity and local equilibrium for the hydrodynamical fluctuations around nonequilibrium steady states, together with an extension of Onsager's regression hypothesis to these states. On this basis, we establish canonical generalisations of both the Onsager reciprocity relations and the Onsager-Machlup fluctuation theory to nonequilibrium steady states, and we show that the spatial correlations of the hydrodynamical fluctuations are generically of long range in these states.

## 1. Introduction

It is an empirical fact that the laws of classical macroscopic continuum mechanics are extremely general and independent, in form, of microscopic constitution. This suggests that it should be possible to base a statistical mechanical treatment of these laws on very general arguments that are centred on macroscopic observables of the hydrodynamical type. In fact, such a treatment was initiated by Onsager [1], in his ground-breaking work of nonequilibrium thermodynamics, which served to relate the macroscopic dynamics of systems close to equilibrium to certain extremely general properties of their underlying microscopic dynamics. This kind of approach to the connection between nonequilibrium thermodynamics and microphysics has been subsequently pursued by the present author in works [2-4] designed to form a bridge between quantum microdynamics and macroscopic quantum continuum mechanics rather than a derivation of the latter from the former\*: its scope, moreover, is not restricted to states close to global equilibrium.

The present article is devoted to an expository account of our quantum macrostatisti-

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\* It is thus radically different from the derivations of Euler, though not Navier-Stokes, hydrodynamics of models of a plasma [5] and of a fermionic system with short range interactions [6] from their underlying quantum dynamics

cal picture of the nonequilibrium thermodynamics of reservoir driven macroscopic systems [3, 4]. This is based on general assumptions of (a) local equilibrium at a certain mesoscopic level, (b) chaoticity of the currents associated with the locally conserved quantum fields and (c) an extension of Onsager's regression hypothesis to fluctuations about nonequilibrium states. On the basis of these hypotheses, which have been substantiated for certain tractable models [7], we have obtained both a nonlinear generalisation of Onsager's theory and the result that the spatial correlations of hydrodynamical observables of reservoir driven open systems are generically of long range in nonequilibrium steady states. This latter result, which had previously been obtained for some special classical stochastic models [8-10], marks a crucial difference between equilibrium and nonequilibrium states, since the spatial correlations carried by the former are of short range, except at critical points.

In Section 2, we present our model of macroscopic quantum systems in general terms, from both the microscopic and the hydrodynamical standpoints. In Section 3, we formulate the fluctuation process executed by the hydrodynamical variables, subject to the above assumptions (a)-(c), and from these we derive canonical generalisations of both the Onsager reciprocity relations and the Onsager-Machlup [11] fluctuation dynamics to nonequilibrium steady states. In Section 4 we show that, under the above assumptions, the spatial correlations of hydrodynamical observables in nonequilibrium steady states are generically of long range. We conclude in Sec. 5 with further brief comments on the theory presented here.

## 2. The Model

We take the model to be a quantum system,  $\Sigma$ , of  $N$  particles located in a bounded open connected region  $\Omega_N$  of a  $d$ -dimensional space,  $X$ , and coupled at its boundary,  $\partial\Omega_N$ , to an array of reservoirs  $\mathcal{R} = \{\mathcal{R}_\alpha\}$ .  $\Sigma$  is thus an open system, comprising part of the composite  $\Sigma^c := (\Sigma + \mathcal{R})$ , which we assume to be conservative. The particle number  $N$  is a variable parameter of the model. We assume that  $\Omega_N$  is the dilation by a factor  $L_N$  of a fixed region  $\Omega$  of unit volume and that both  $\Omega$  and the mean particle number density,  $\nu$ , of  $\Sigma$  are  $N$ -independent. Thus  $\Omega_N = \{L_N x | x \in \Omega\}$  and  $\nu L_N^d = N$ . Throughout this article we employ units in which  $\hbar$  and  $k_{Boltzmann}$  are equal to unity.

### 2.1. The Quantum Statistical Picture.

We employ the standard operator algebraic description [12-14, 2] of conservative system such as  $\Sigma^c$  or  $\Sigma$ , in the situation where it is isolated from  $\mathcal{R}$ . Thus, we recall that, in this description, the observables of such a system are the self-adjoint elements of a  $*$ -algebra,  $\mathcal{A}$ , its states,  $\omega$ , are linear, positive, normalised, expectation functionals  $A \rightarrow \langle \omega : A \rangle$  on  $\mathcal{A}$ , and its dynamics, in the Heisenberg picture, corresponds to a one-parameter group of automorphisms  $A \rightarrow A_t$  of  $\mathcal{A}$ . We assume that all its interactions are invariant under space translations, rotations and time reversals. The equilibrium states of the system at inverse temperature  $\beta$  are characterised by the Kubo-Martin-Schwinger (KMS) condition [15, 2]:-

$$\langle \omega : A_t B \rangle = \langle \omega ; B A_{t+i\beta} \rangle, \quad \forall A, B \in \mathcal{A}.$$

The set of states satisfying this condition is convex, and its extremal elements correspond to pure thermodynamic phases [16, 2].

We assume that  $\Sigma$  has a set of extensive, conserved observables  $\hat{Q} = (\hat{Q}_1, \dots, \hat{Q}_n)$ , which intercommute up to surface corrections and are *thermodynamically complete* in the sense that the equilibrium states corresponding to pure phases are labelled by the expectation values of the global density of  $\hat{Q}$  in the limit  $N \rightarrow \infty$  [2]. We denote by  $s(q)$  the equilibrium entropy density for which this expectation value is  $q = (q_1, \dots, q_n)$ . The function  $s$  is concave and may be formulated by standard statistical mechanical procedures [12, 2]. The thermodynamical control variable conjugate to  $q$  is then  $\theta = (\theta_1, \dots, \theta_n)$ , with  $\theta_r = \partial s(q) / \partial q_r$ . Thus, assuming that  $\hat{Q}_1$  is the energy observable,  $\theta_1$  is the inverse temperature. We restrict our considerations to situations where the system is confined to a single phase region for which  $q$  lies in a connected domain,  $\Delta$ , of  $\mathbf{R}^n$ , in which the function  $s$  is smooth and the correspondence between  $q$  and  $\theta$  is one-to-one. It follows that the Hessian matrix  $s''(q) := [\partial^2 s / \partial q_k \partial q_l]$  is invertible and we define

$$J(q) := -s''(q)^{-1}. \quad (2.1)$$

Further, by the one-to-one correspondence between  $q$  and  $\theta$ , the equilibrium states in this phase may equivalently be labelled by the latter or the former. We assume the symmetries of time inversions, space translations and space rotations are unbroken in this pure equilibrium phase.

*The Reservoirs.* We assume that each reservoir  $\mathcal{R}_\alpha$  has a thermodynamically complete set,  $\hat{Q}_\alpha$ , of extensive conserved observables  $\hat{Q}_\alpha = (\hat{Q}_{\alpha,1}, \dots, \hat{Q}_{\alpha,n})$ , which are the natural counterparts of the  $\hat{Q}_r$ 's and which satisfy the condition that the  $\Sigma - \mathcal{R}_\alpha$  interactions conserve each  $(\hat{Q}_r + \hat{Q}_{\alpha,r})$ . The thermodynamical control variable,  $\theta_\alpha$ , of  $\mathcal{R}_\alpha$  is then the canonical counterpart of the variable  $\theta$  of  $\Sigma$ . We denote by  $\omega_\alpha(\theta_\alpha)$  the equilibrium state of  $\mathcal{R}_\alpha$  corresponding to  $\theta_\alpha$ .

*The Steady State of  $\Sigma^c$ .* We assume that the system  $\Sigma$  and the reservoirs  $\{\mathcal{R}_\alpha\}$  are independently prepared, in the remote past, with  $\Sigma$  in an arbitrary state  $\phi$  and each  $\mathcal{R}_\alpha$  in its equilibrium state  $\omega_\alpha(\theta_\alpha)$ , and that the reservoirs are then coupled to spatially disjoint regions of  $\partial\Omega_N$ , whose union comprises that surface. Then, under rather general conditions [17, 18],  $\Sigma^c$  evolves to a terminal state  $\omega^c$ , which evidently depends on the variables  $\{\theta_\alpha\}$ . In the special case where the  $\theta_\alpha$ 's for the different reservoirs are all equal,  $\omega^c$  is a canonical equilibrium state. Otherwise, it is the nonequilibrium steady state of  $\Sigma^c$  for the specified conditions.

*The Local Density of  $\hat{Q}$ .* We assume that the extensive conserved observables  $\hat{Q}$  of  $\Sigma$  have locally conserved, position dependent densities  $\hat{q}(x) = (\hat{q}_1(x), \dots, \hat{q}_n(x))$  with associated currents  $\hat{j}(x) = (\hat{j}_1(x), \dots, \hat{j}_n(x))$ . More precisely, we assume that the local conservation law for  $\hat{q}$  prevails for the evolution of this field not only for the dynamics of  $\Sigma$ , when isolated, but also for that of the composite system  $\Sigma^c = (\Sigma + \mathcal{R})$ . Thus, denoting by  $\hat{q}_t(x) \equiv \hat{q}(x, t)$  and  $\hat{j}_t(x) \equiv \hat{j}(x, t)$  the evolutes of the fields  $\hat{q}$  and  $\hat{j}$ , respectively, for the

system  $\Sigma^c$ , these time- dependent fields and currents satisfy the local conservation law

$$\frac{\partial \hat{q}_t}{\partial t} + \nabla \cdot \hat{j}_t = 0.$$

For simplicity, we assume that both the extensive observable  $\hat{Q}$  and its position dependent density  $\hat{q}$  are invariant under time reversals, i.e. velocity reversals.

In accordance with the general requirements of quantum field theory [19], we assume that the fields  $\hat{q}_t$  and  $\hat{j}_t$  are operator valued distributions\*.

## 2.2. The Hydrodynamical Description.

We assume that the hydrodynamical picture is given by a continuum mechanical law governing the evolution of an  $n$ -component, locally conserved classical field  $q_t(x) = (q_{1,t}(x), \dots, q_{n,t}(x))$ , on a macroscopic space-time scale that we shall presently specify. As we shall see in Section 2.3, this field represents a rescaled expectation value of the quantum field  $\hat{q}_t(x)$ , but here we shall be concerned with just its phenomenological properties.

We assume that the current  $j_t = (j_{1,t}, \dots, j_{n,t})$  associated with  $q_t$  satisfies a constitutive equation of the form

$$j_t(x) = \mathcal{J}(q_t : x) \quad (2.2)$$

where  $\mathcal{J}$  is a functional of the field  $q_t$  and the position  $x$ . Consequently, by local conservation,  $q_t$  evolves according to an autonomous law,

$$\frac{\partial}{\partial t} q_t(x) = \mathcal{F}(q_t; x) \equiv - \nabla \cdot \mathcal{J}(q_t; x), \quad (2.3)$$

subject to boundary conditions that are fixed by the reservoirs\*\*. We assume that the resultant  $q_t(x)$  is confined to the single phase region  $\Delta$ , introduced in Sec. 2.1. For simplicity, we base our explicit treatment here on the case of nonlinear diffusions where

$$\mathcal{J}(q; x) = -K(q_t(x)) \nabla q_t(x); \quad \mathcal{F}(q_t; x) = \nabla \cdot (K(q_t(x)) \nabla q_t(x)), \quad (2.4)$$

and where  $K(q)$  is an  $n$ -by- $n$  matrix  $[K_{kl}(q)]$  and acts by matrix multiplication on  $\nabla q$ . This evolution is therefore invariant under scale transformations  $x \rightarrow \lambda x$ ,  $t \rightarrow \lambda^2 t$ .

We choose the unit of length for the hydrodynamic scale to be  $L_N$ . Thus, in view of our stipulation that  $\Omega_N = L_N \Omega$ , the region occupied by  $\Sigma$  in the hydrodynamical picture is  $\Omega$ . Furthermore, the scale invariance noted after Eq. (2.4) signifies that a length scale  $L_N$

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\* This assumption may easily be seen to be valid in standard physical cases. For example, the field representing the position dependent particle number density is  $\sum_{r=1}^N \delta(x - x_r)$ , where  $x_r$  is the position of the  $r$ 'th particle.

\*\* As discussed in [4], these conditions serve to equate the spatial boundary value of the local control variable  $\theta_t(x) := s'(q_t(x))$  with that of the reservoir  $\mathcal{R}_\alpha$  that is in contact with  $\Sigma$  at  $x$ .

corresponds to a time scale  $L_N^2$ . We assume that there is just one hydrodynamically steady state of the system, as represented by the field  $q(x)$  and the current  $j(x)$  for which  $\mathcal{F}(q; x) = 0$  and  $j(x) = \mathcal{J}(q; x)$ . To lighten the notation, we define

$$K_q := K \circ q; \quad J_q := J \circ q; \quad \tilde{K}_q := K_q J_q. \quad (2.5)$$

*Hydrodynamical Perturbations of the Steady State.* By Eq. (2.3), the linearised equations of motion for ‘small’ perturbations  $\delta q_t$  and  $\delta j_t$  of the steady state field and current  $q(x)$  and  $j(x)$ , respectively, are

$$\delta j_t(x) = \mathcal{K} \delta q_t(x) := \frac{\partial}{\partial \lambda} \mathcal{J}(q + \delta q_t; x)|_{\lambda=0}; \quad \frac{\partial}{\partial t} \delta q_t(x) = \mathcal{L} \delta q_t(x) := \frac{\partial}{\partial \lambda} \mathcal{F}(q + \delta q_t; x)|_{\lambda=0}. \quad (2.6)$$

We note that, since  $q_t$  and  $j_t$  are rescaled expectation values of the distribution valued quantum fields  $\hat{q}_t$  and  $\hat{j}_t$ , it follows that they too are distributions. Specifically, denoting by  $\mathcal{D}(\Omega)$  and  $\mathcal{D}_V(\Omega)$  the L. Schwartz spaces [19] of real, infinitely differentiable scalar and vector valued functions, respectively, on  $\Omega$  with support in the interior of that region, we take  $\delta q_t$  and  $\delta j_t$  to be elements of the duals,  $\mathcal{D}'^n(\Omega)$  and  $\mathcal{D}'_V^n(\Omega)$ , of the  $n$ -fold tensorial powers,  $\mathcal{D}^n(\Omega)$  and  $\mathcal{D}_V^n(\Omega)$ , of  $\mathcal{D}(\Omega)$  and  $\mathcal{D}_V(\Omega)$ , respectively. We denote elements of  $\mathcal{D}^n(\Omega)$  and  $\mathcal{D}_V^n(\Omega)$  by  $f = (f_1, \dots, f_n)$  and  $g = (g_1, \dots, g_n)$ , respectively, and we equip these latter spaces with inner products  $(\cdot, \cdot)$  and  $(\cdot, \cdot)_V$ , respectively, defined by the formulae

$$(f, f') = \sum_{r=1}^n \int_{\Omega} dx f_r(x) f'_r(x); \quad (g, g')_V = \sum_{r=1}^n \int_{\Omega} dx g_r(x) \cdot g'_r(x). \quad (2.7)$$

According to the above definitions,  $\mathcal{L}$  is a linear transformation of  $\mathcal{D}'^n(\Omega)$  and  $\mathcal{K}$  is a linear mapping of  $\mathcal{D}'^n(\Omega)$  into  $\mathcal{D}'_V^n(\Omega)$ . By Eqs. (2.3) and (2.6), these linear operators are related by the equation

$$\mathcal{L} = -\nabla \cdot \mathcal{K}. \quad (2.8)$$

We assume that  $\mathcal{L}$  is the generator of a one-parameter semigroup  $\{T_t | t \in \mathbf{R}_+\} = T(\mathbf{R}_+)$  of linear transformations of  $\mathcal{D}'^n(\Omega)$ , and we denote the duals of  $\mathcal{L}$  and  $T_t$  by  $\mathcal{L}^*$  and  $T_t^*$ , respectively. Thus, by Eq. (2.6),

$$\delta q_t = T_{t-s} \delta q_s; \quad \delta j_t = -\mathcal{K} T_{t-s} \delta q_s \quad \forall t \geq s. \quad (2.9)$$

### 2.3. Connection between the Quantum and Hydrodynamical Pictures.

In order to bring our formulation of the quantum field  $\hat{q}_t(x)$  and current  $\hat{j}_t(x)$  into line with their hydrodynamical counterparts, we rescale the position  $x$  and the time  $t$  by the factors  $L_N$  and  $L_N^2$ , respectively. The resultant quantum field  $\tilde{q}_t(x)$  and current  $\tilde{j}_t(x)$  are therefore given by the formulae

$$\tilde{q}_t(x) = \hat{q}(L_N x, L_N^2 t); \quad \tilde{j}_t(x) = L_N \hat{j}(L_N x, L_N^2 t). \quad (2.10)$$

Thus,  $\tilde{q}_t$  and  $\tilde{j}_t$  satisfy the local conservation law

$$\frac{\partial \tilde{q}_t}{\partial t} + \nabla \cdot \tilde{j}_t = 0.$$

We assume that, in general, the phenomenological fields  $q_t$  and  $j_t$  of Sec. 2.2 are just the expectation values of  $\tilde{q}_t$  and  $\tilde{j}_t$ , respectively, for the prevailing state of  $\Sigma^c$  in the limit where  $N$  tends to infinity. In particular, the stationary field  $q(x)$  and current  $j(x)$  are the limiting values, as  $N$  tends to infinity, of  $\langle \omega^c; \tilde{q}(x) \rangle$  and  $\langle \omega^c; \tilde{j}(x) \rangle$ , respectively.

*The Hydrodynamical Fluctuation Fields.* We represent the fluctuations of the hydrodynamic field  $\tilde{q}_t$  and current  $\tilde{j}_t$  about their means for the steady state  $\omega^c$  by the fluctuation field  $\tilde{\xi}_t = (\tilde{\xi}_{1,t}, \dots, \tilde{\xi}_{n,t})$  and its associated current  $\tilde{\eta}_t = (\tilde{\eta}_{1,t}, \dots, \tilde{\eta}_{n,t})$ , as defined by the formulae

$$\tilde{\xi}_t(x) = N^{1/2}(\tilde{q}_t(x) - \langle \omega^c; \tilde{q}_t(x) \rangle); \quad \tilde{\eta}_t(x) = N^{1/2}(\tilde{j}_t(x) - \langle \omega^c; \tilde{j}_t(x) \rangle), \quad (2.11)$$

the factor  $N^{1/2}$  being canonical for fluctuations [20]. It follows immediately from these definitions that the local conservation law for  $\tilde{q}_t$  implies the corresponding one for  $\tilde{\xi}_t$ , which we express in the following integral form.

$$\tilde{\xi}_t - \tilde{\xi}_s + \tilde{\zeta}_{t,s} = 0 \quad \forall t, s \in \mathbf{R}, \quad (2.12)$$

where

$$\tilde{\zeta}_{t,s} = \int_s^t du \nabla \cdot \tilde{\eta}_u. \quad (2.13)$$

### 3. The Fluctuation Process.

We denote by  $\tilde{\xi}_t(f)$  and  $\tilde{\zeta}_{t,s}(g)$  the time-dependent smeared fields obtained by integrating  $\tilde{\xi}_t$  and  $\tilde{\zeta}_{t,s}$  against test functions  $f$  ( $\in \mathcal{D}^n(\Omega)$ ) and  $g$  ( $\in \mathcal{D}_V^n(\Omega)$ ), respectively. The statistical properties of these smeared fields in the steady state  $\omega^c$  are then represented by the correlation functions given by the expectation values, for this state, of the monomials in the  $\tilde{\xi}_t(f)$ 's and the  $\tilde{\zeta}_{t,s}(g)$ 's. We assume that these functions are continuous in all their arguments (the test functions  $f, g$  and the time variables  $s, t$ ) and converge, as  $N \rightarrow \infty$ , to corresponding finite valued ones for a stochastic process executed by fields  $(\xi_t(f), \zeta_{t,s}(g))$ , respectively: conditions for the validity of these assumptions are specified in [4]. Then, in view of the completeness of the Schwartz  $\mathcal{D}$ -spaces, the correlation functions for the resultant process are continuous in the test functions  $f$  and  $g$  and measurable in the time variables. Further, under general space-time asymptotic abelian conditions on the fields  $(\tilde{\xi}_t(f), \tilde{\zeta}_{t,s}(g))$ , the process is classical [4]. Thus, assuming these conditions, the quantum process  $(\tilde{\xi}, \tilde{\zeta})$  converges *in law*, as  $N$  tends to infinity, to the classical process  $(\xi, \zeta)$ . It follows immediately that  $\xi$  and  $\zeta$  satisfy the natural counterpart of the local conservation law (2.12), which when integrated against a test function  $f$  ( $\in \mathcal{D}^n(\Omega)$ ) takes the form

$$\xi_t(f) - \xi_s(f) = \zeta_{t,s}(\nabla f). \quad (3.1)$$

### 3.1. The Regression Hypothesis.

We assume a natural generalisation of Onsager's [1] regression hypothesis to the effect that the fluctuation field  $\xi_t$  regresses according to the same dynamical law as the weak externally induced perturbations  $\delta q_t$  of the stationary hydrodynamical field  $q$ . Thus, since the latter law is given by Eq. (2.9), we take our regression hypothesis to be that

$$E(\xi_t|\xi_s) = T_{t-s}\xi_s \quad \forall t \geq s, \quad (3.2)$$

where  $E(\cdot|\xi_s)$  denotes the conditional expectation given the field  $\xi_s$  at time  $s$ . Hence, denoting the static two-point function by

$$W(f, f') := E(\xi(f)\xi(f')), \quad (3.3)$$

it follows from Eq. (3.2) and the stationarity of the  $\xi$ -process that

$$E(\xi_t(f)\xi_s(f')) = W(T_{t-s}^*f, f') \quad \forall f, f' \in \mathcal{D}^n(\Omega), \quad t, s(\leq t) \in \mathbf{R}. \quad (3.4)$$

We further develop the relation between fluctuations and externally induced hydrodynamical perturbations by noting that, by Eq. (2.6),  $\int_s^t du \delta j_u = \int_s^t du \mathcal{K} \delta q_u$ . Correspondingly, we designate the *secular* part of the time-integrated current fluctuation  $\zeta_{t,s}$  to be  $\int_s^t du \mathcal{K} \xi_u$ , and we define the remainder of  $\zeta_{t,s}$  to be its *stochastic* part, namely

$$\zeta_{t,s}^{stoc} = \zeta_{t,s} - \int_s^t du \mathcal{K} \xi_u. \quad (3.5)$$

It follows then from Eqs. (2.8) and (3.5) that the local conservation law (3.1) is equivalent to the following equation.

$$\xi_t(f) - \xi_s(f) - \int_s^t du \xi_u(\mathcal{L}^*f) = \zeta_{t,s}^{stoc}(\nabla f). \quad (3.6)$$

Since  $\mathcal{L}$  is the generator of  $T(\mathbf{R}_+)$ , it follows from this equation, after some manipulation [4], that the two-point function of  $\zeta^{stoc}$  is related to that of  $\xi$  according to the formula

$$\begin{aligned} E(\zeta_{t,s}(\nabla f)\zeta_{t',s'}(\nabla f')) &= -[W(\mathcal{L}^*f, f') + W(f, \mathcal{L}^*f')][[s, t] \cap [s', t']] \\ &\quad \forall f, f' \in \mathcal{D}^n(\Omega), \quad t, s, t', s' \in \mathbf{R}, \end{aligned} \quad (3.7)$$

where the last factor is the length of the intersection of the intervals  $(s, t)$  and  $(s', t')$ .

### 3.2. The Chaoticity Hypothesis.

We assume that the stochastic current is chaotic in the sense that the space-time correlations of the unsmeared stochastic field  $\zeta_{t,s}^{stoc}$  are of short range on the microscopic scale and therefore, since  $L_N$  tends to infinity with  $N$ , of zero range on the hydrodynamic

scale. Further, in accordance with the central limit theorem for fluctuation fields with short range correlations [21], we assume that the process  $\zeta^{stoc}$  is Gaussian. Thus our chaoticity assumption is that

(C.1) the process  $\zeta^{stoc}$  is Gaussian; and

(C.2)  $E(\zeta_{t,s}^{stoc}(g)\zeta_{t',s'}(g')) = 0$  if either  $(s, t) \cap (s', t')$  or  $\text{supp}(g) \cap \text{supp}(g')$  is empty.

The following proposition was inferred\* in [4] from this assumption and Schwartz's compact and point support theorems [20, Ths. 26,35].

**Proposition 3.1.** *Under the assumption (C.2), the two-point function for  $\zeta$  takes the form*

$$E(\zeta_{t,s}(g)\zeta_{t',s'}(g')) = Z(g, g')|[s, t] \cap [s', t']| \quad \forall g, g' \in \mathcal{D}^n(\Omega), \quad t, s, t', s' \in \mathbf{R}, \quad (3.8)$$

where  $Z$  is an element of  $\mathcal{D}'^n(\Omega) \otimes \mathcal{D}'^n(\Omega)$  with support in the domain  $\{(x, x) | x \in \Omega\}$ .

The following corollary to this proposition follows immediately from a comparison of Eqs. (3.7) and (3.8).

**Corollary 3.2.** *Under the same assumptions,*

$$Z(\nabla f, \nabla f') = -[W(\mathcal{L}^* f, f') + W(f, \mathcal{L} f')] \quad \forall f, f' \in \mathcal{D}^n(\Omega). \quad (3.9)$$

### 3.3. The Local Equilibrium Hypothesis.

For an *equilibrium* state, the static two point function for the field  $\xi$  has been derived from the KMS condition in the following form [2, Ch. 7, Appendix C].

$$W_{eq}(f, f') \equiv E_{eq}(\xi_{eq}(f)\xi_{eq}(f')) = (f, J_q f') \quad \forall f, f' \in \mathcal{D}^n(\Omega), \quad (3.10)$$

where the subscript *eq*, here and elsewhere, refers to the equilibrium state and the function  $J_q$  and the inner product  $(\cdot, \cdot)$  and are defined in Eqs. (2.1) and (2.7), respectively. Further, the process  $\xi_{eq}$  inherits from its underlying quantum dynamics the property of invariance under time reversals. Hence, in view of the stationarity of this process, it follows from a standard argument [4] that

$$W_{eq}(\mathcal{L}_{eq}^* f, f') = W_{eq}(\mathcal{L}_{eq}^* f', f) \quad \forall f, f' \in \mathcal{D}^n(\Omega). \quad (3.11)$$

Turning now to the process  $\zeta^{stoc}$ , we see that, by translational invariance of the equilibrium state, it follows from by Eqs. (2.4)-(2.6) that  $\mathcal{L}_{eq} = K_q \Delta$  and thence, by Eqs. (2.7), (3.9) and (3.10), that

$$Z_{eq}(\nabla f, \nabla f') = (\nabla f, [\tilde{K}_q + \tilde{K}_q^*] \nabla f')_V, \quad (3.12)$$

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\* In fact the proof in [4] invoked the supplementary assumption that the l.h.s of Eq. (3.8) was continuous in its time variables. However, that assumption is redundant, since the required continuity can be derived by exploiting the positivity of that formula for  $f = f', t = t', s = s'$ .



where  $\tilde{K}_q^\star$  is the transpose of the matrix  $\tilde{K}_q$ . Further, a simple argument based on the rotational symmetry of the equilibrium state leads to an extension of this formula to the following one [4].

$$Z_{eq}(g, g') = (g, [\tilde{K}_q + \tilde{K}_q^\star]g')_V \quad \forall g, g' \in \mathcal{D}_V^n(\Omega). \quad (3.13)$$

Eqs. (3.10), (3.11) and (3.13) are our equilibrium conditions for the fluctuation process. We now formulate the local properties of these conditions in terms of test functions that are highly localised around an arbitrary point  $x_0$  of  $\Omega$ . Thus, for  $x_0 \in \Omega$  and sufficiently small  $\epsilon \in \mathbf{R}_+$ , we define the transformations  $f \rightarrow f_{x_0, \epsilon}$  of  $\mathcal{D}^n(\Omega)$  and  $g \rightarrow g_{x_0, \epsilon}$  of  $\mathcal{D}_V^n(\Omega)$  by the formulae

$$f_{x_0, \epsilon}(x) = \epsilon^{-d/2} f(\epsilon^{-1}(x - x_0)); \quad g_{x_0, \epsilon}(x) = \epsilon^{-d/2} g(\epsilon^{-1}(x - x_0)). \quad (3.14)$$

Thus, since by Eqs. (2.4)-(2.6),  $\mathcal{L}_{eq} = K_q \Delta$  and thus  $\epsilon^2 \mathcal{L}_{eq}^\star f_{x_0, \epsilon} = [\mathcal{L}^\star f]_{x_0, \epsilon}$ , it follows that the equilibrium conditions (3.10), (3.11) and (3.13) are equivalent to the following ones, which evidently represent properties at the (hydrodynamical) point  $x_0$  in the limit  $\epsilon \rightarrow 0$ .

$$W_{eq}(f_{x_0, \epsilon}, f'_{x_0, \epsilon}) = (f, J_q f') \quad \forall f, f' \in \mathcal{D}^n(\Omega), \quad x_0 \in \Omega, \quad (3.15)$$

$$\epsilon^2 W_{eq}(\mathcal{L}_{eq}^\star f_{x_0, \epsilon}, f'_{x_0, \epsilon}) = \epsilon^2 W_{eq}(\mathcal{L}_{eq}^\star f'_{x_0, \epsilon}, f_{x_0, \epsilon}) \quad \forall f, f' \in \mathcal{D}^n(\Omega), \quad x_0 \in \Omega, \quad (3.16)$$

both sides of this equation being equal to  $(\mathcal{L}_{eq}^\star f, J_q f')$ ; and

$$Z_{eq}(g_{x_0, \epsilon}, g'_{x_0, \epsilon}) = (g, [\tilde{K}_q + \tilde{K}_q^\star]g')_V \quad \forall g, g' \in \mathcal{D}_V^n(\Omega), \quad x_0 \in \Omega \quad (3.17)$$

Correspondingly, we take our local equilibrium conditions for the nonequilibrium steady state to be given by removing the subscript  $eq$ , replacing  $J_q$  and  $K_q$  in these formulae by  $J_q(x_0)$  and  $K_q(x_0)$ , respectively, and then passing to the limit  $\epsilon \rightarrow 0$ . Thus, the resultant conditions are

$$\lim_{\epsilon \downarrow 0} W(f_{x_0, \epsilon}, f'_{x_0, \epsilon}) = (f, J_q(x_0) f') \quad \forall f, f' \in \mathcal{D}^n(\Omega), \quad x_0 \in \Omega, \quad (3.18)$$

$$\lim_{\epsilon \downarrow 0} \epsilon^2 W(\mathcal{L}^\star f_{x_0, \epsilon}, f'_{x_0, \epsilon}) = \lim_{\epsilon \downarrow 0} \epsilon^2 W(\mathcal{L}^\star f'_{x_0, \epsilon}, f_{x_0, \epsilon}) \quad \forall f, f' \in \mathcal{D}^n(\Omega), \quad x_0 \in \Omega, \quad (3.19)$$

and

$$\lim_{\epsilon \downarrow 0} Z(g_{x_0, \epsilon}, g'_{x_0, \epsilon}) = (g, [\tilde{K}_q(x_0) + \tilde{K}_q^\star(x_0)]g')_V \quad \forall g, g' \in \mathcal{D}_V^n(\Omega), \quad x_0 \in \Omega. \quad (3.20)$$

Further, by Eq. (3.8), the chaoticity assumption (C.2) implies that  $Z(g, g')$  vanishes if the supports of  $g$  and  $g'$  are disjoint. Consequently, by Schwartz's compact and point support theorems [20, Ths. 26, 35], Eq. (3.20) implies that [4]

$$Z(g, g') = (g, [\tilde{K}_q + \tilde{K}_q^\star]g')_V \quad \forall g, g' \in \mathcal{D}_V(\Omega). \quad (3.21)$$

Hence, by Eq. (3.7),

$$E(\zeta_{t,s}^{stoc}(g) \zeta_{t',s'}(g')) = (g, [\tilde{K}_q + \tilde{K}_q^\star]g')_V | [s, t] \cap [s', t'] | \quad \forall g, g' \in \mathcal{D}_V^n(\Omega), \quad t, s, t', s' \in \mathbf{R}. \quad (3.22)$$

### 3.4. Generalised Reciprocity Relations and Onsager-Machlup Processes.

The following proposition was proved in [4].

**Proposition 3.3.** *Assuming the regression hypotheses and local equilibrium conditions (3.18) and (3.19), the  $n$ -by- $n$  matrix-valued function  $\tilde{K}_q$  on  $\Omega$  is symmetric. This constitutes a nonequilibrium, position-dependent generalisation of the Onsager reciprocity relations.*

In order to show that the Onsager-Machlup theory extends to the present setting, we define

$$w_t(f) = \zeta_{t,0}^{stoc}(\nabla f) \quad \forall f \in \mathcal{D}^n(\Omega), \quad t \in \mathbf{R} \quad (3.23)$$

and infer from this formula, together with Eq. (3.22), Prop. 3.1 and the Gaussian condition (C.1) that  $w$  is the Wiener process whose two-point function is

$$E([w_t(f) - w_s(f)][w_{t'}(f') - w_{s'}(f')]) = 2(\nabla f, \tilde{K}_q \nabla f')_V \quad \forall f, f' \in \mathcal{D}^n(\Omega)_V, \quad t, s, t', s' \in \mathbf{R}. \quad (3.24)$$

The following proposition then ensues immediately from Eqs. (3.6), (3.23) and (3.24).

**Proposition 3.4.** *Assuming the regression, chaoticity and local equilibrium hypothesis, the  $\xi$ -field executes a generalised Onsager-Machlup process given by the Langevin equation*

$$d\xi_t = \mathcal{L}\xi_t dt + dw_t, \quad (3.25)$$

where  $w$  is the Wiener process specified above.

### 4. Long Range Correlations of the $\xi$ -Process.

The next proposition and the subsequent comment signify that, under the above assumptions, the spatial correlations of the  $\xi$  field are generically of non-zero range on the macroscopic scale and hence of long (infinite!) range on the microscopic one.

**Proposition 4.1.** *Let  $\Phi_q$  and  $\Psi_q$  be the  $n$ -by- $n$  matrix valued scalar and vector fields, respectively, in  $\Omega$  defined by the formula*

$$\Phi_q = \Delta \tilde{K}_q + \nabla \cdot \Psi_q \quad (4.1)$$

and

$$\Psi_{q;k,l}(x) = \sum_{k',l'=1}^n \left[ \frac{\partial}{\partial q_{l'}(x)} K_{k,k'}(q(x)) \right] [J_{l,l'}(q(x)) \nabla q_{k'}(x) - J_{k',l}(q(x)) \nabla q_{l'}(x)]. \quad (4.2)$$

*Then, under the assumptions of the regression, chaoticity and local equilibrium hypotheses, a sufficient condition for the static spatial correlations of the  $\xi$ -process to be of non-zero range on the macroscopic scale is that either  $\Phi_q$  does not vanish or that the matrix  $\Psi_q$  is symmetric.*

**Comment.** The functions  $\Phi_q$  and  $\Psi_q$  are determined by the forms of the entropy function  $s$ , the transport coefficient  $K$  and the form of  $q|_{\partial\Omega}$ , which represents the boundary conditions imposed by the reservoirs on the profile of the function  $q$ . Since these functions are mutually independent, it follows from Prop. 4.1 that it is only under very special conditions that the spatial correlations of the  $\xi$ -field are of zero range on the hydrodynamic scale. In other words, these correlations are generically of non-zero range on that scale and consequently of infinite range on the microscopic one.

**Example.** In the case of the simple exclusion process, whether in the classical picture [8-10] or the quantum one [7],  $n = 1$ ,  $d = 1$ ,  $K(q) = 1$ ,  $s(q) = -q\log(q) - (1-q)\log(1-q)$  and  $q(x) = a + bx$ , where  $a$  and  $b$  are constants, the latter being non-zero in nonequilibrium steady states. Then it follows from Eqs. (2.1), (4.1) and (4.2) that, in these states,  $\Psi_q = 0$  and  $\Phi_q(x) = -2b^2 \neq 0$ . Hence, long (infinite) range correlations prevail in this model.

**Proof of Prop. 4.1.** We employ a *reductio ad absurdum* argument. Thus we start by assuming that the spatial correlations of the  $\xi$ -process are of zero range, on the hydrodynamic scale, i.e. that the support of  $W$  lies in the domain  $\{x, x' \in \Omega | x = x'\}$ . Then by pursuing the argument used to obtain Eq. (3.21), we see that this assumption, together with the local equilibrium condition (3.18), implies that

$$W(x, x') = J_q(x)\delta(x - x'). \quad (4.3)$$

On the other hand, Eq. (3.21) and Prop. 3.3 imply that Eq. (3.9) is equivalent to the following equation for the distribution  $W$ .

$$[\mathcal{L} + \mathcal{L}']W(x, x') = 2\nabla \cdot (\tilde{K}_q(x)\nabla\delta(x - x')), \quad (4.4)$$

where  $\mathcal{L}'$  is the version of  $\mathcal{L}$  that acts on functions of  $x'$ . On inserting this formula (4.3) for  $W$  into this equation and using Eqs. (2.4)-(2.6), it follows, after some manipulation, that

$$\Phi_q(x)\delta(x - x') + [\Psi_q(x) - \Psi_q^*(x)]\delta'(x - x') = 0.$$

Thus, the assumption that the spatial correlations of the  $\xi$ -process are of zero range on the hydrodynamic scale cannot be sustained unless  $\Phi_q$  vanishes and  $\Psi_q$  is symmetric.

## 5. Concluding Remarks.

In this work, the crucial links between quantum microdynamics and classical macroscopic continuum mechanics are provided by the mesoscopic picture of hydrodynamical fluctuations. Apart from generalising both the Onsager reciprocity relations and the Onsager-Machlup fluctuation process to nonequilibrium steady states, the ensuing theory establishes, on a very general basis, that the spatial correlations of the hydrodynamical variables are generically of infinite range in these states. This result, which had previously been established for certain classical stochastic models [8-10], marks a crucial difference between equilibrium and nonequilibrium steady states, since the spatial correlations of the former are of short range, except at critical points [22].

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